University of North Georgia
Department of Mathematics

Instructor: Berhanu Kidane

Course: Precalculus Math 1113

Text Books: For this course we use free online resources:

See the folder Educational Resources in Shared class files

1) http://www.stitz-zeager.com/szca07042013.pdf (**Book1**)
2) Trigonometry by Michael Corral (**Book 2**)

Other online resources:

E– Book: http://msenux.redwoods.edu/IntAlgText/

Tutorials: http://www.wtamu.edu/academic/anns/mps/math/mathlab/col_algebra/index.htm

- http://archives.math.utk.edu/visual.calculus /
- http://www.ltcconline.net/greenl/java/index.html
- http://en.wikibooks.org/wiki/Trigonometry
- Animation Lessons: http://flashytrig.com/intro/teacherintro.htm
- http://www.sosmath.com/trig/trig.html

For more free supportive educational resources consult the syllabus

Trigonometric Graphs (Book 2 Page 103)
Periodic Functions

A **periodic function** is a function that repeats its values in regular intervals or periods. The most important examples are the trigonometric functions, which repeat over intervals of length 2π radians. Periodic functions are used throughout science to describe oscillations, waves, and other phenomena that exhibit periodicity.

Definition (Periodic Function & Period page 109 Book 2)

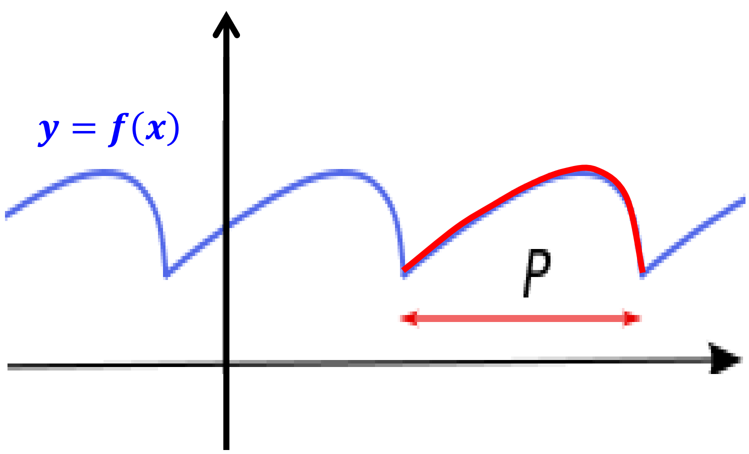
A function f is said to be **periodic** if there is a number $\mathbf{P > 0}$ such that, $f(x+P) = f(x)$ **for all values of** x. If there exist a **least positive** constant P with this property, we call that number the **prime period or** simply the **Period** of the function.

A function with period P will **repeat** its values on **intervals** of **length** P, and these intervals are sometimes also referred to as **periods**. Geometrically, a function f is periodic with period P if the **graph** of f is **invariant** under **translation** in the ***x*-direction** by a distance of P.

Examples: Graphs of Some Periodic Functions

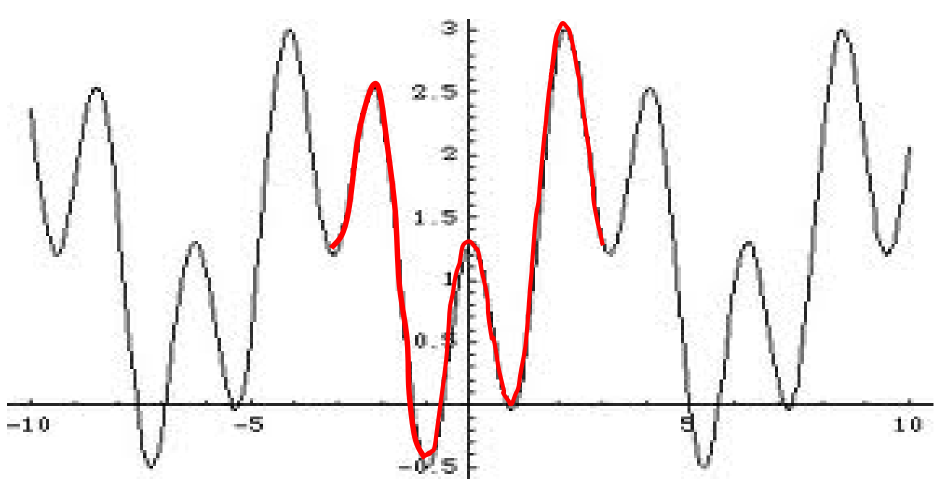
The **colored curves** in the graphs are the **periods** of the functions

a)



$y = f(x)$ is a function of **period** (**Prime period**) P

b)

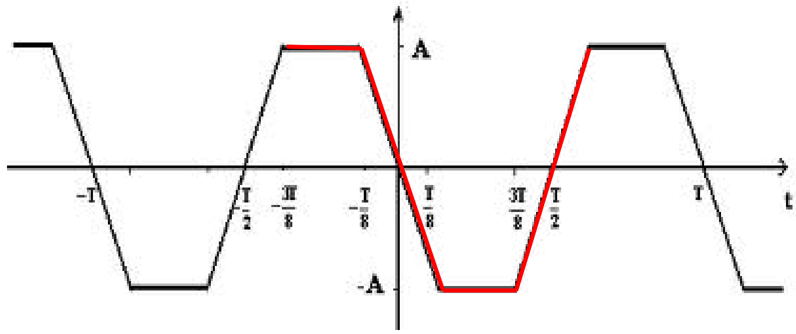


This is a periodic function of period 6

Definition (Amplitude)

The **maximum displacement** from the **equilibrium position** is called **Amplitude**

Example:



This is a periodic function of period $\frac{3}{4}\pi$ and **Amplitude A**

Note that for all six trigonometric functions it holds that:

1) $\sin x = \sin(x + 2\pi)$
2) $\cos x = \cos(x + 2\pi)$
3) $\tan x = \tan(x + 2\pi)$
4) $\csc x = \csc(x + 2\pi)$
5) $\sec x = \sec(x + 2\pi)$
6) $\cot x = \cot(x + 2\pi)$

Thus all **six trigonometric** functions **are periodic**

Graphs of Sine and Cosine

Notice that the graphs of sine and cosine repeat their values in a regular fashion, i.e. the **sine** and **cosine functions** are **periodic**. Using the sum difference it can easily be verified that:

$\sin(x + 2n\pi) = \sin x$ and $\cos(x + 2n\pi) = \cos x$, for all **integers n**,

The least positive integer for which the equations hold is 2π that is:

$\sin(x + 2\pi) = \sin x$ and $\cos(x + 2\pi) = \cos x.$

The number 2π is called the **prime period or period** of the **sine** and the **cosine** function. We are saying the **sine** and **cosine** functions are **periodic functions** of **period** and **2π.**

We can use graphing utilities to graph the **sine** and **cosine functions; or tables of special angles to sketch the graphs of the sine and the cosine functions in an interval of length 2π**

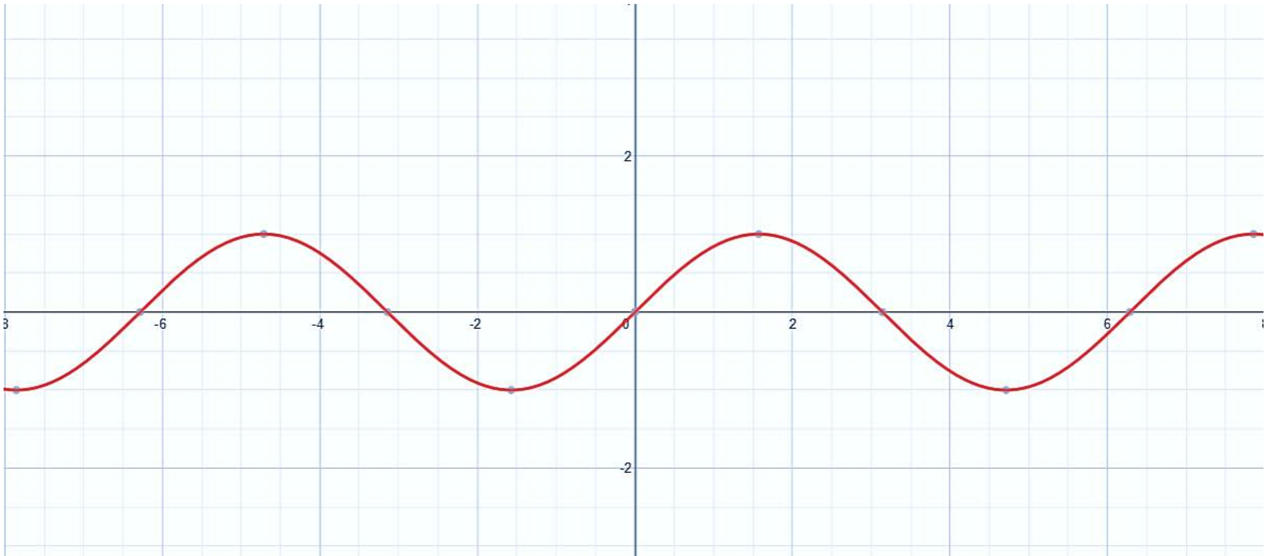
Complete the following tables (give exact values)

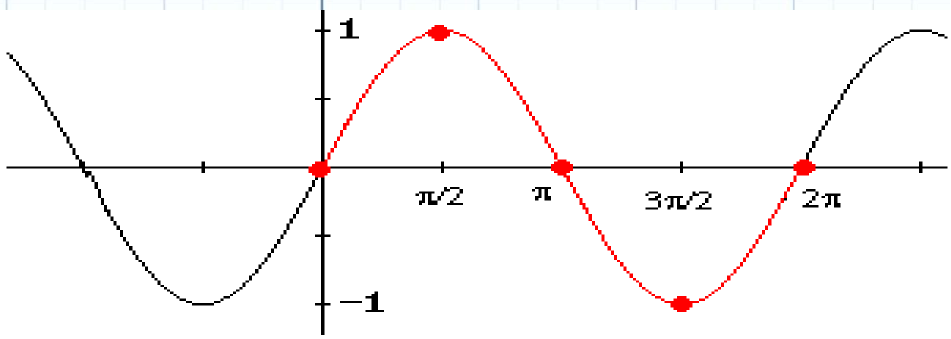
x	$\sin x$	x	$\sin x$
0	0	π	0
$\pi/6$	1/2	$5\pi/6$	-1/2
$\pi/4$	$1/\sqrt{2}$	$5\pi/4$	$-1/\sqrt{2}$
$\pi/3$	$\sqrt{3}/2$	$4\pi/3$	$-\sqrt{3}/2$
$\pi/2$	1	$3\pi/2$	-1
$2\pi/3$	$\sqrt{3}/2$	$5\pi/3$	$-\sqrt{3}/2$
$3\pi/4$	$1/\sqrt{2}$	$11\pi/6$	-1/2
$5\pi/6$	1/2	2π	0

x	$\cos x$	x	$\cos x$
0	1	π	-1
$\pi/6$	$\sqrt{3}/2$	$5\pi/6$	$-\sqrt{3}/2$
$\pi/4$	$1/\sqrt{2}$	$5\pi/4$	$-1/\sqrt{2}$
$\pi/3$	1/2	$4\pi/3$	-1/2
$\pi/2$	0	$3\pi/2$	0
$2\pi/3$	-1/2	$5\pi/3$	1/2
$3\pi/4$	$-1/\sqrt{2}$	$11\pi/6$	$\sqrt{3}/2$
$5\pi/6$	$-\sqrt{3}/2$	2π	1

Graphs using graphing utilities

1) The Sine function : $f(x) = sin(x)$

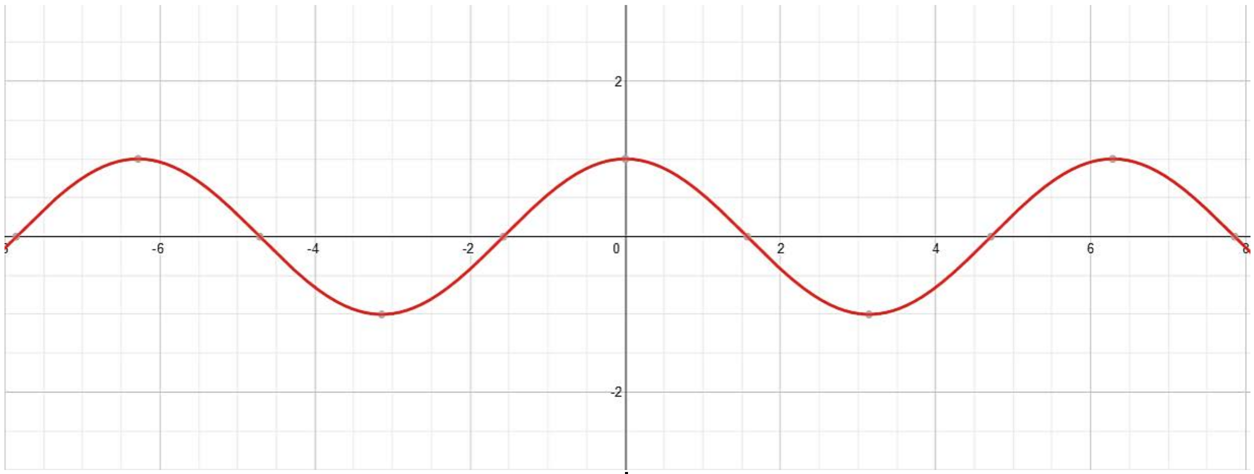


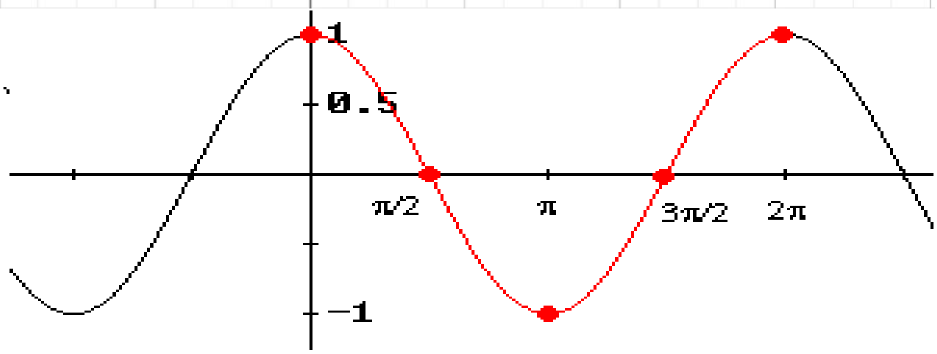


Properties

- **Domain:** The set of all real numbers
- **Range:** $[-1, 1]$
- **Amplitude:** A = 1
- **Period:** $P = 2\pi$, see graph, indicated in **red**
- x **intercepts:** $x = k\pi$, where k is an integer.
- y **intercepts:** $y = 0$
- **Maximum points:** $\left(\frac{\pi}{2} + 2k\pi , 1\right)$, where k is an integer.
- **Minimum points:** $\left(\frac{3\pi}{2} + 2k\pi , -1\right)$, where k is an integer.
- **symmetry:** since $sin(-x) = -sin(x)$, $sin(x)$ is an **odd function** and its graph is **symmetric with respect** to the origin $(0, 0)$.
- **Intervals of increase/decrease:** over one **period**, 0 *to* 2π, $sin(x)$ is **increasing** on the **intervals** $\left(0, \frac{\pi}{2}\right)$ and $\left(\frac{3\pi}{2}, 2\pi\right)$, and **decreasing** on the **interval** $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$.

2) **The Cosine Function:** $f(x) = cos\ (x)$





Properties

- **Domain:** The set of all real numbers
- **Range:** $[-1\ , 1]$
- **Amplitude:** **A = 1**
- **Period:** $P = 2\pi$, see graph, indicated in **red**
- x **intercepts:** $x = \pi/2 + k\,\pi$, where k is an integer.
- **y intercepts:** $y = 1$
- **Maximum points:** $(2k\,\pi\ ,\ 1)$, where k is an integer.
- **Minimum points:** $(\pi + 2k\pi\ ,\ -1)$, where k is an integer.
- **Symmetry:** since $cos(-x) = cos\ (x)$, $cos\ (x)$ is an **even function** and its **graph** is **symmetric with respect** to the y – axis.
- **Intervals of increase/decrease:** over one period, $0\ to\ 2\pi$; $cos\ (x)$ is **decreasing** on $(0\ ,\ \pi)$ and is **increasing** on $(\pi\ ,\ 2\pi)$.

Graphs of Transformations of Sine and Cosine Functions (Page)

Recall Transformations of the Plane

Translations, **Reflections, Stretches** and **Shrinks**

Translations

1) **Vertical Translation**: $y = f(x) \pm c, for\ c > 0$

The graph of $y = f(x) + c$ is the graph of $y = f(x)$ shifted vertically **c** units up

The graph of $y = f(x) + c$ is the graph of $y = f(x)$ shifted vertically **c** units down

2) **Horizontal Translations:** $y = f(x \pm c), for\ c > 0$

The graph of $y = f(x - c)$ is the graph of $y = f(x)$ shifted horizontally **c** units to the right

The graph of $y = f(x - c)$ is the graph of $y = f(x)$ shifted horizontally **c** units to the left.

Reflections

1) **Across the x-axis:**

The graph of $y = -f(x)$ is the **reflection** of the graph of $y = f(x)$ across the **x-axis**.

2) **Across the y-axis:**

The graph of $y = f(-x)$ is the **reflection** of the graph of $y = f(x)$ across the **y-axis**.

Stretches and Shrinks

1) Vertical Stretching and shrinking

To graph $y = cf(x)$:

If $c > 1$, **stretch** the graph of $y = f(x)$ **vertically** by a **factor of c**

If $0 < c < 1$, **shrink** the graph of $y = f(x)$ **vertically** by a **factor of c**

2) Horizontal Stretching and shrinking

To graph $y = f(cx)$:

If $c > 1$, **shrink** the graph of $y = f(x)$ **horizontally** by a **factor of $1/c$**

If $0 < c < 1$, **stretch** the graph of $y = f(x)$ **horizontally** by a **factor of $1/c$**

Example 0: Sketch the graph using the square function

a) $f(x) = x^2 - 1$

b) $f(x) = (2x)^2 - 1$

c) $f(x) = 2(x^2 - 1)$

d) $f(x) = 2(x - 1)^2 - 3$

e) $f(x) = \left(\frac{1}{2}x\right)^2 - 1$

f) $f(x) = \frac{1}{2}(x^2 - 1)$

g) $f(x) = -(x - 1)^2 + 2$

h) $f(x) = -(2x - 1)^2 + 2$

Example 1: Sketch the graph from sine and cosine curves

a) $f(x) = -\cos x$

b) $f(x) = 1 + \sin s$

c) $f(x) = -2\cos x$

d) $f(x) = 2 + \sin(x + 1)$

e) $f(x) = -\sin x$

f) $f(x) = 1 - \cos x$

g) $f(x) = 2\sin x$

h) $f(x) = 1 - \cos\left(x - \frac{1}{2}\right)$

Sine and Cosine Curves

In this section we consider **sine** and **cosine functions** (curves) defined by:

1) $f(x) = a\sin(kx)$ or $f(x) = a\cos(kx)$, $k > 0$
2) $f(x) = a\sin[k(x-b)]$ **or** $f(x) = a\cos[k(x-b)]$, $k > 0$ **(Shifted Sine** and **Cosine** Curves)

Let P be the **period** of the function $f(x) = a\sin[k(x-b)]$ and b be any real number

By definition of the period of a function, $f(x+P) = f(x)$, which is the same as saying

$$a\sin[k(x+P-b)] = a\sin[k(x-b)]$$

$$a\sin[k(x-b)+kP] = a\sin[k(x-b)]$$

Since the **sine function** is **periodic** of **period** 2π, it must hold that $kP = 2\pi$, which implies the period of the **sine** function $f(x) = a\sin[k(x-b)]$ **is** $P = \frac{2\pi}{k}$.

Similarly the period of the **cosine** function $f(x) = a\cos[k(x-b)]$ is $P = \frac{2\pi}{k}$

Thus we have the following statements justified:

1) **The sine** and **cosine** curves: $y = a\sin(kx)$ and $y = a\cos(kx)$, $k > 0$ has **amplitude** $|\, a \,|$ and **period** $2\pi/k$

An **appropriate interval** to **sketch graphs of** $y = a\sin(kx)$ and $y = a\cos(kx)$ on one complete period is $\left[0,\ \frac{2\pi}{k}\right]$.

Example 2: Find the **amplitude** and **period** of each function, and **sketch** the **graph**.

a) $f(x) = -2\sin(3x)$

b) $f(x) = -1 + \cos 2x$

c) $f(x) = 3\cos\left(\frac{1}{2}x\right)$

d) $f(x) = 1 + 2\sin 3x$

2) The **shifted Sine** and **Cosine** curves**;** $y = a\sin\big(k(x-b)\big)$ and $y = a\cos(k(x-b))$ $k > 0$, has **amplitude** $|\, a \,|$ and **period** $2\pi/k$ and **phase shift** $\mathbf{b}$.

An **appropriate interval** to **sketch graphs of** $y = a\sin(k(x-b))$ and $y = a\cos(k(x-b))$ on one complete period is $\left[b,\ b + \frac{2\pi}{k}\right]$.

Example 3: Find the **amplitude** and **period** of each function, and **sketch** the **graph**.

a) $f(x) = -2\sin(3(x-\pi/4))$

b) $f(x) = \frac{4}{3}\cos\big(2(x+\pi/3)\big)$

c) $f(x) = \frac{3}{2}\sin(2.5x + 3\pi)$

Reading Exerciase

Examples: 5.4 page 109, 5.5 page 111, and 5.6 & 5.7 page 112

Note: Let $f(t) = A\sin bt + B\cos bt$ where **A, B** and **b** are constants and $A \neq 0$. We can write $f(t)$ in the form

$$f(t) = \sqrt{A^2+B^2}\left(\frac{A}{\sqrt{A^2+B^2}}\sin bt + \frac{B}{\sqrt{A^2+B^2}}\cos bt\right) \qquad \textbf{(1)}$$

Let **c** be a real number such that $\tan c = \frac{B}{A}$, and $\cos c = \frac{A}{\sqrt{A^2+B^2}}$ and $\sin c = \frac{B}{\sqrt{A^2+B^2}}$, then we have

$$f(t) = \sqrt{A^2+B^2}(\cos c\ \sin bt + \sin c\ \cos bt) \text{ and}$$

Observe $\cos c\ \sin bt + \sin c\ \cos bt = \sin(bt + c)$**.** If we let $a = \sqrt{A^2+B^2}$ then **(1) can be written as** $f(t) = a\sin(bt + c)$**.** Thus proving the statement below.

Theorem: If **t** is any real number**, A, B** and **b** are constants and $A \neq 0$**,** then

$A\sin bt + B\cos bt = a\sin(bt + c)$, where $a = \sqrt{A^2+B^2}$, and **c** satisfies $\cos c = \frac{A}{\sqrt{A^2+B^2}}$ and $\sin c = \frac{B}{\sqrt{A^2+B^2}}$

Example 4: Let $f(t) = 2\sqrt{3}\sin t + 2\cos t$

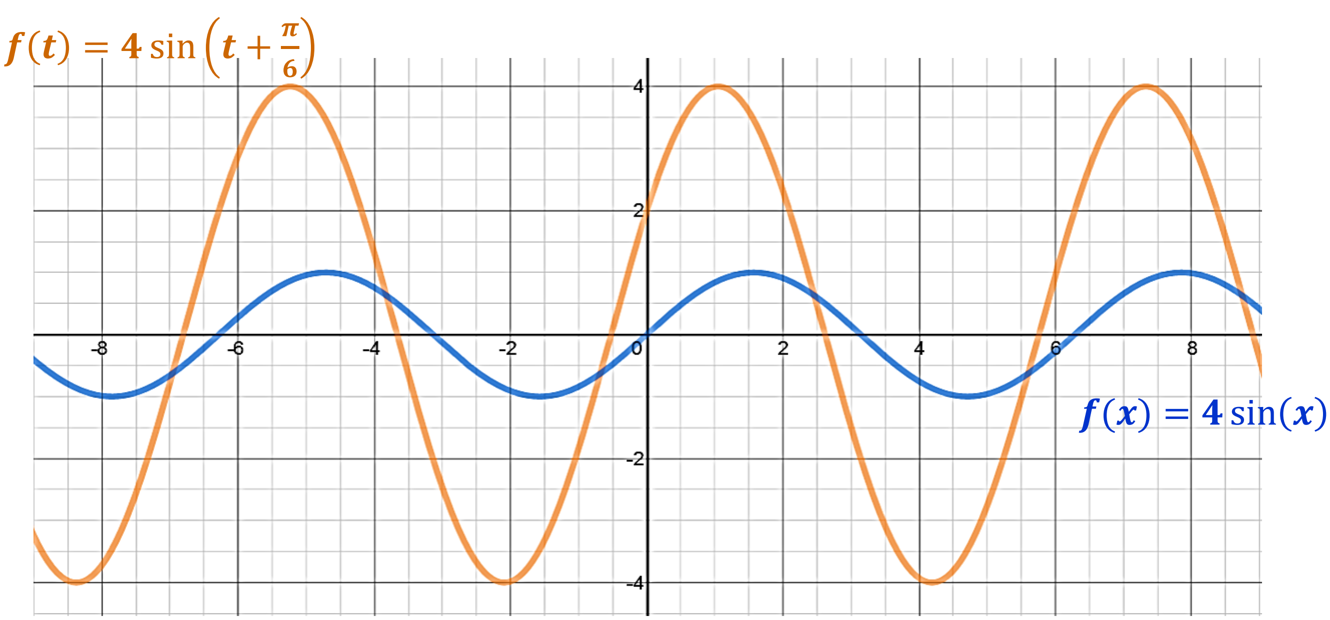
a) Write $f(t)$ in an equation of the form $f(t) = a\sin(bt + c)$
b) Determine the **amplitude**, **period** and **phase shift** of f, and **sketch the graph**

Solution: $B = 2$, $A = 2\sqrt{3}$, $b = 1$

$a = \sqrt{A^2+B^2} = 4$**; and so,** $\sin c = \frac{A}{\sqrt{A^2+B^2}} = \frac{2}{4} = \frac{1}{2}$**, and** $\cos c = \frac{B}{\sqrt{A^2+B^2}} = \frac{\sqrt{3}}{2}$ implies $c = \frac{\pi}{6}$

So, a) $f(t) = 4\sin\left(t + \frac{\pi}{6}\right)$

b) $Amplitude = 4$, $period = 2\pi$ and $Phase\ shift = -\pi/6$

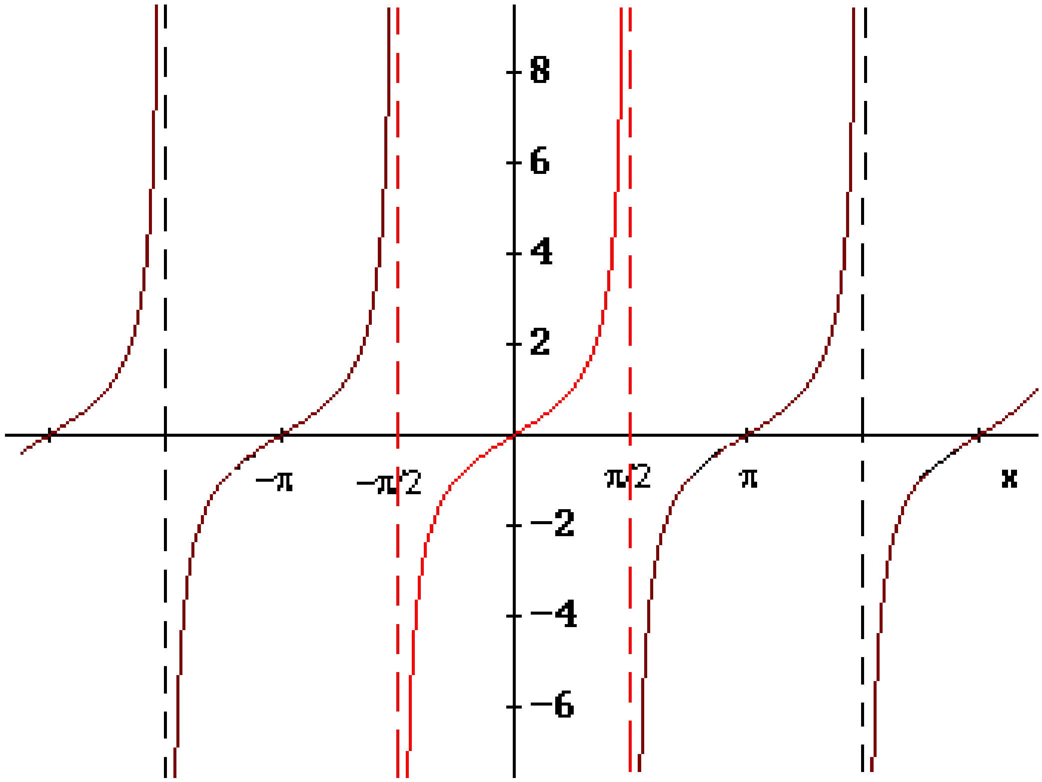


Reading Exerciase
Examples: 5.9 page 114, 5.10 page 115, and 5.11 & 5.12 page 117
Practice Problems (Book 2)
Page 118 & 119 # 1 – 12, 14, 15, 19 – 25.

More Trigonometric Graphs

3) The Tangent Function: $f(x) = tan\ (x)$

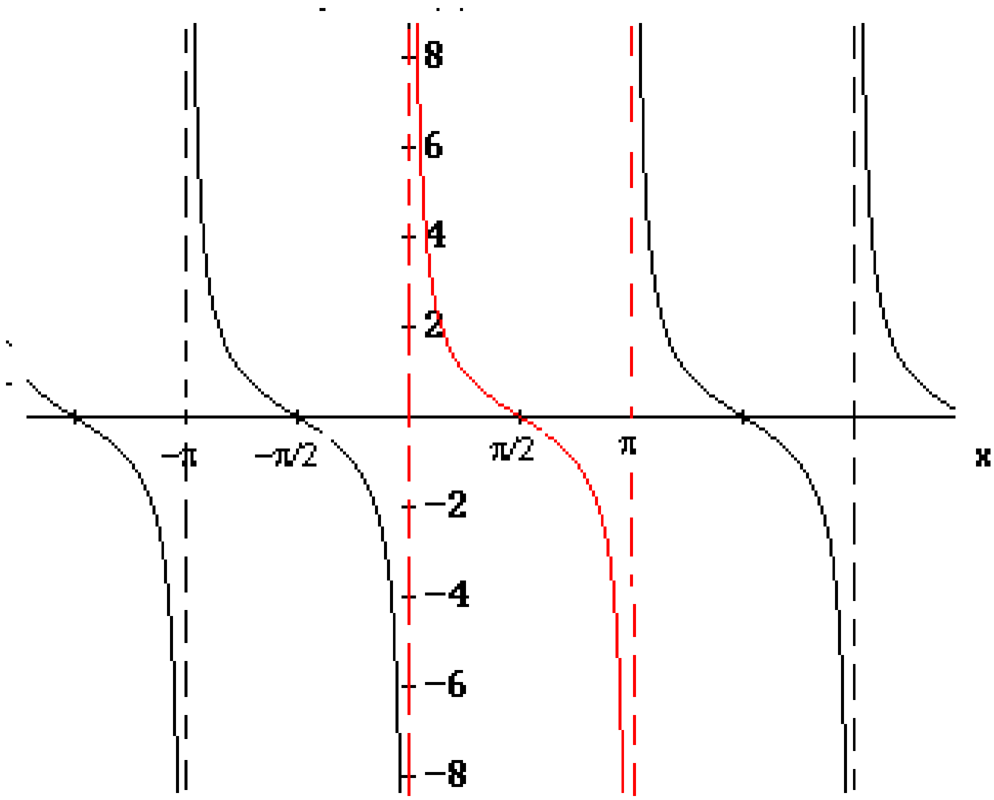
- **Graph**



- **Domain:** The set of all real numbers except $\pi/2 + k\pi$, k is an integer.
- **Range:** The set of all real numbers
- **Period:** $P = \pi$, see graph indicated in **red**
- ***x* intercepts:** $x = k\pi$, where k is an integer.
- **y intercepts:** $y = 0$
- **Symmetry:** since $tan(-x) = -tan(x)$ then $tan\ (x)$ is an **odd function** and its graph is symmetric with respect the origin.
- **Intervals of increase/decrease:** over one period and from $-\pi/2\ to\ \pi/2$, $tan\ (x)$ is increasing.
- **Vertical asymptotes:** $x = \pi/2 + k\pi$, where k is an integer.

4) **The Cotangent Function:** $f(x) = cot\ (x)$

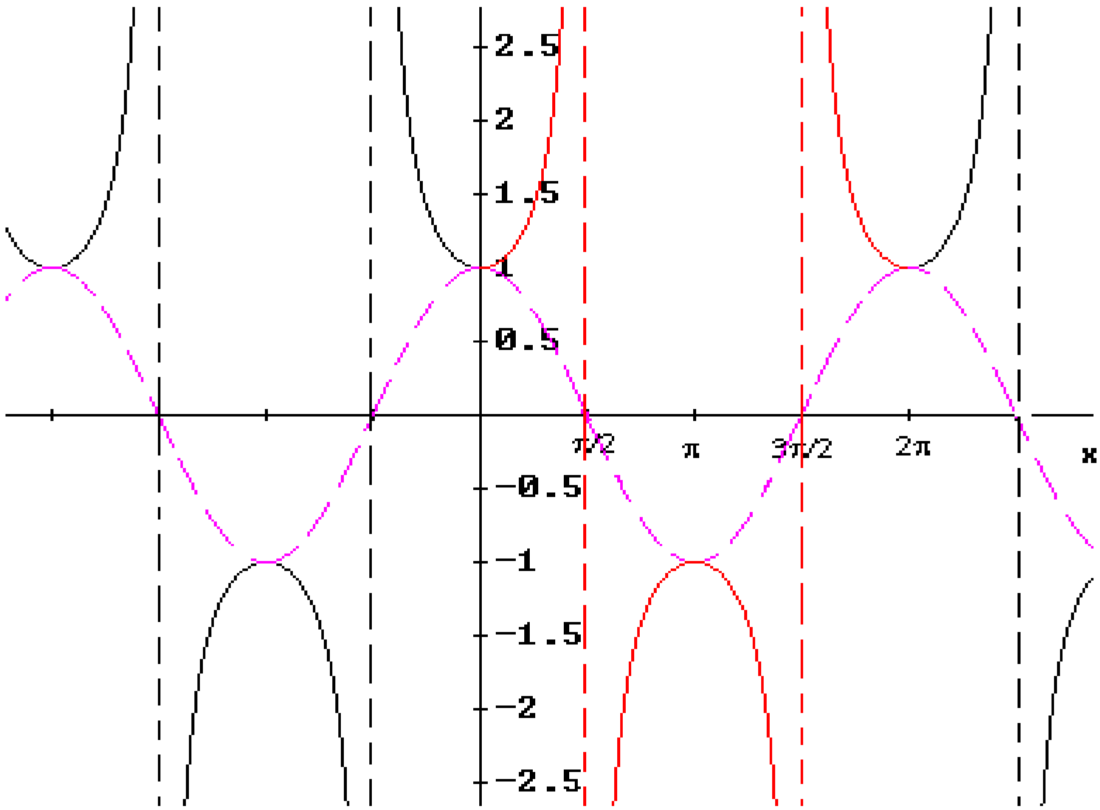
- **Graph**



- **Domain:** The set of all real numbers except $k\pi$, k is an integer.
- **Range:** The set of all real numbers
- **Period** $P = \pi$, see graph indicated in **red**
- ***x* intercepts:** $x = \pi/2 + k\pi$, where k is an integer.
- **No *y* - intercept**
- **Symmetry:** since $cot(-x) = -\ cot(x)$, $cot\ (x)$ is an **odd function** and its graph is symmetric with respect the origin.
- **Intervals of increase/decrease:** over one period and from **0** to π, $cot\ (x)$ is decreasing.
- **Vertical asymptotes:** $x = k\pi$, where k is an integer.

5) **The Secant Function:** $f(x) = sec\,(x)$

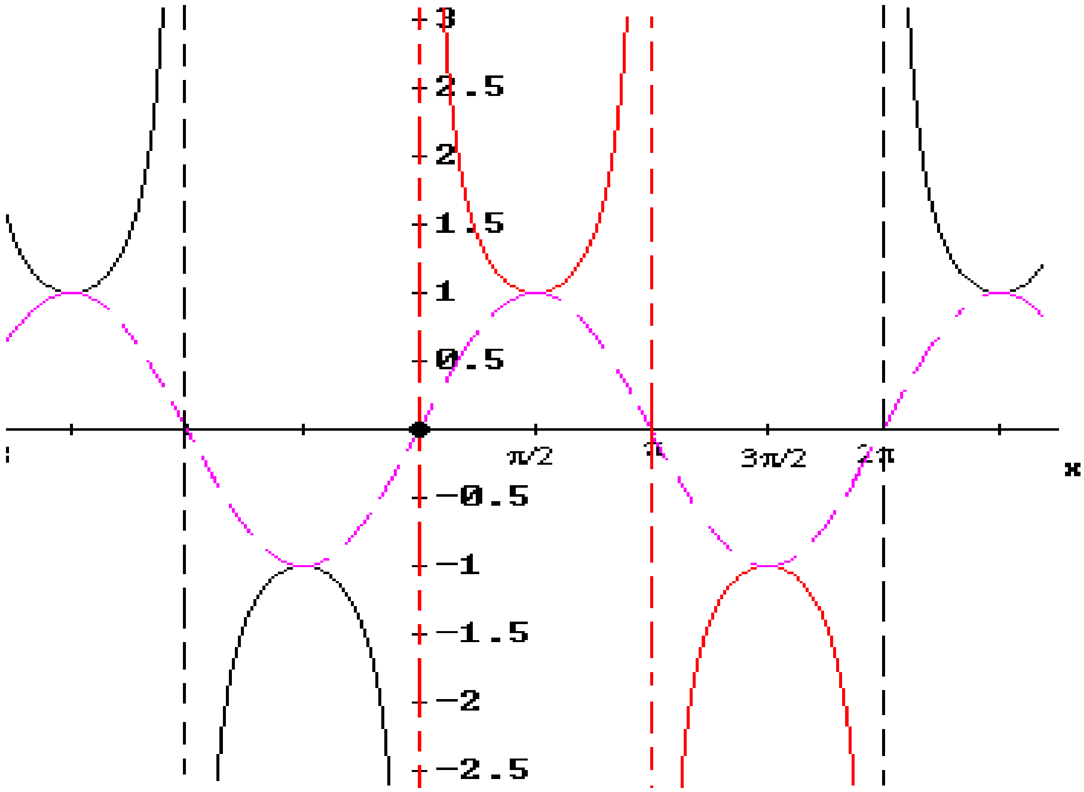
- **Graph**



- **Domain:** The set of all real numbers except $\pi/2 + k\pi$, n is an integer.
- **Range:** $(-\infty, -1] \cup [1, +\infty)$
- **Period** $P = 2\pi$, see graph indicated in **red**
- **No x - intercept**
- **y intercepts:** **y = 1**
- **Symmetry:** since $sec(-x) = sec\,(x)$, $sec\,(x)$ is an **even** function and its graph is symmetric with respect to the y axis.
- **Intervals of increase/decrease:** over one period and from 0 *to* 2π, $sec\,(x)$ is increasing on $(0\,,\ \pi/2) \cup (\pi/2\,,\ \pi)$ and decreasing on $(\pi\,,\ 3\pi/2) \cup (3\pi/2\,,\ 2\pi)$.
- **Vertical asymptotes:** $x = \frac{\pi}{2} + k\pi$, where k is an integer.

6) **The Cosecant Function:** $f(x) = csc(x)$

- **Graph**



- **Domain:** The set of all real numbers except $k\pi$, k is an integer.
- **Range:** $(-\infty, -1] \cup [1, +\infty)$
- **Period:** $P = 2\pi$, see graph indicated in red
- **No x and y - intercept**
- **Symmetry:** since $csc(-x) = -csc(x)$, $csc(x)$ is an **odd** function and its graph is symmetric with respect the origin.
- **Intervals of increase/decrease:** over one period and from 0 *to* 2π, $csc(x)$ is **decreasing** on $(0, \pi/2) \cup (3\pi/2, 2\pi)$ and **increasing** on $(\pi/2, \pi) \cup (\pi /, 3\pi/2)$.
- **Vertical asymptotes:** $x = k\pi$ where k is an integer.